A Technique for the Experimental Determination of the Electrical Conductivity of Plasmas

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A technique is presented which permits the measurement of the electrical conductivity of a plasma by means of observing a diminution of radio frequency magnetic flux through the plasma. This techique has found application in the measurement of conductivity of stationary plasmas, flowing steady-state plasmas, and plasmas produced in the electromagnetic shock tube. The rudiments of the theory of operation of a device using this technique are presented, and a measurement of plasma conductivity is described for the case of the hot exhaust gas of a deflagrating sample of solid rocket fuel expanding in a vacuum.

I. Introduction

THE electrical conductivity of an ionized gas is of interest in energy conversion studies in which electrical energy is to be extracted from a stationary or a flowing gas. In electrical propulsion or in arcjet applications, the electrical conductivity of the plasma is a good indication of the efficiency of the plasma generation process. Also, the conductivity of the exhaust gases of a rocket motor can be related to the gases' average temperature; hence, it is useful in heat-transfer studies and materials' tests.

Various techniques have been proposed for the direct measurement of the conductivity of a plasma. Lin, Resler, and Kantrowitz[1] presented a technique which, applied to the moving plasma in a shock tube, related the displacement of the magnetic flux of a large coil by the conductive gas to the conductivity of the gas. The use of this technique was subsequently reported by Lamb and Lin,[2] and Rosa.[3] Another technique was developed by Blackman,[4] who observed the change in Q of a short solenoid coupled to an rf oscillator when a plasma passed through its volume, and who related this change to the plasma conductivity. Variations of this technique[5-11] which removed some of its restrictions and placed it on a firm theoretical foundation[6, 7, 9] were later reported.

Most of the techniques referenced previously are adequate for their intended application within certain limits. In the case of the devices that rely on a "dragging," or a displacement of the magnetic flux, the plasma must have a reasonably high conductivity, and it must be in motion. The devices based on Blackman's technique can function well with either a stationary or moving plasma, but require the use of sensitive frequency or rf power measuring equipment. It has been reported[5, 6] that, at low levels of plasma conductivity, these devices fail to yield meaningful data due to 1) the neglect of the effect of displacement currents in the underlying theory and 2) interaction of the axial electric field of the coil with the coaxial plasma. Apparently, this failure is enhanced when a highly polar electrolyte is substituted for the plasma. Lary and Olson,[7] however, have recently reported that these effects do not appear when the coil is immersed in the electrolyte and have also reported the development of a very interesting immersible conductivity probe. This probe apparently disturbs the plasma, as evidenced by its reported rate of heating, and hence may be strongly influencing that which it is supposed to measure in all but free-molecular flow situations.

A relatively simple technique for the measurement of the electrical conductivity of a uniform plasma has been developed in the course of a study[5, 12] of the electrodeless discharge in gases. This technique utilizes a device that has the advantages of simplicity, theoretical justification, ease of calibration, and adaptability to new experimental situations. The device is inexpensive and within the fabrication capabilities of any laboratory. It has found application in the measurement of the conductivity of stationary plasmas,[5] flowing steady-state plasmas,[12] in the measurement of the time-dependent conductivity in the shocked gas within the electromagnetic shock tube,[13] and in the measurement of the conductivity of the exhaust of a laboratory-sized rocket motor using solid fuel.

Briefly, the device operates as follows: A plasma is situated within the volume of a long solenoid, which carries a weak rf current. The plasma can be in motion or stationary. Eddy currents are induced in the plasma according to Faraday's law of induction, and the resultant decrease in magnetic flux in the plasma volume is experimentally observed. This quantity is then related to the plasma conductivity through the skin-depth relation.

II. Analysis

The behavior of an alternating magnetic field **B** in a medium of uniform conductivity σ, permeability μ_0, and permittivity ϵ_0, and no net free charge, is described by the wave equation

$$\nabla^2\mathbf{B} + (\omega^2\mu_0\epsilon_0 - i\sigma\mu_0\omega)\mathbf{B} = 0 \tag{1}$$

where ω is the angular frequency of the field.

Within the volume of a long solenoid, the magnetic field can be considered entirely axial and spatially uniform. Then, when a cylindrical conductor, such as an axial plasma, is brought into this volume, the wave equation becomes

$$\frac{d^2B_z}{dr^2} + \frac{1}{r}\frac{dB_z}{dr} + i^3\sigma\mu_0\omega B_z = 0 \tag{2}$$

where it has been assumed that $\sigma/\omega\epsilon_0 \gg 1$. The solution of Eq. (2) is

$$B_z(r) = C_1J_0[(i^3\sigma\mu_0\omega)^{1/2}r] + C_2Y_0[(i^3\sigma\mu_0\omega)^{1/2}r] \tag{3}$$

where J_0 and Y_0 are Bessel functions of the zeroth order and, respectively, the first and second kinds. If the uniform magnetic field has the magnitude B_0 at the outer radius R of the conductor, and is finite along the axis (where $r = 0$), Eq. (3) is evaluated as follows:

$$B_z(r) = B_0J_0[(i^3\sigma\mu_0\omega)^{1/2}r]/J_0[(i^3\sigma\mu_0\omega)^{1/2}R] \tag{3a}$$

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The magnetic flux Φ penetrating the cylindrical conductor is obtained from the surface integral of the magnetic field

$$\Phi = \int_0^R B_z(r) 2\pi r dr \tag{4}$$

Substitution in the argument of the Bessel functions $\delta = (2/\sigma\mu_0\omega)^{1/2}$, where δ is the electromagnetic "skin-depth" in the material, and integration over the plasma cross section, yields an expression for the magnetic flux through the plasma

$$\Phi = 2\pi R^2 B_0 \left(\frac{\delta}{2^{1/2}R}\right) \frac{M_1(2^{1/2}R/\delta)}{M_0(2^{1/2}R/\delta)} e^{i(\theta_1 - \theta_0 - 3\pi/4)} \tag{5}$$

where the M's are the moduli of the first- and zeroth-order Bessel functions with complex argument and are defined (Ref. 14, p. 141) by the following relations:

$$\begin{aligned} M_\nu(z)e^{\pm i\theta_\nu(z)} &= J_\nu(zi^{\pm 3/2}) \\ &= (ber_\nu{}^2 z + bei_\nu{}^2 z)^{1/2}[\cos\theta_\nu(z) \pm i \sin\theta_\nu(z)] \\ \theta_\nu(z) &= \tan^{-1}[bei_\nu z/ber_\nu z] \end{aligned}$$

A voltage is impressed upon a loop of wire wound about the plasma coaxial with the solenoid. This voltage is proportional to the time rate of change of the magnetic flux. The alternating magnetic flux is measured by observing the output of such a loop of wire after an electronic integration. Measurement of the solenoid current I then yields the value of B_0 from the relation

$$B_0 = \mu_0 NI/l \tag{6}$$

a relation valid for a long solenoid with N/l turns per unit length.

Application of Eq. (6) to Eq. (5) with the experimentally determined magnetic flux permits a solution for the quantity $2^{1/2}R/\delta$, from which the plasma conductivity can be readily obtained.

A much easier technique, however, is to determine the magnetic flux relative to some arbitrary conductivity reference value. A calibration is performed in which the magnitude of the magnetic flux Φ_0 is observed with a known conductor through the sensor coil. This value is compared with the magnitude of the magnetic flux observed under similar electrical conditions, but with a coaxial plasma. Then,

$$\left|\frac{\Phi}{\Phi_0}\right| = \frac{\delta}{\delta_0} \frac{M_1(2^{1/2}R/\delta)}{M_0(2^{1/2}R/\delta)} \frac{M_0(2^{1/2}R/\delta_0)}{M_1(2^{1/2}R/\delta_0)} \tag{7}$$

If the reference conductor is a vacuum or air (i.e., $\sigma = 0$), Eq. (7) can be written

$$\left|\frac{\Phi}{\Phi_0}\right| = 2\left(\frac{\delta}{2^{1/2}R}\right) \frac{M_1(2^{1/2}R/\delta)}{M_0(2^{1/2}R/\delta)} \tag{8}$$

In most cases, the sensor coil is physically larger than its conductive core. In this case, the observed flux ratio is Φ'/Φ'_0, where the primed flux values include the contributions from the area between the sensor coil and its conductive core. It is easy to show that, for the initially uniform field, the areal correction takes the form

$$\frac{\Phi}{\Phi_0} = \frac{R_c{}^2}{R^2}\left(\frac{\Phi'}{\Phi_0'} - 1\right) + 1 \tag{9}$$

where R_c is the radius of the sensor coil, R is the radius of its core, Φ'/Φ_0' is the observed flux ratio, and Φ/Φ_0 is that ratio which would have been observed had the sensor coil and the conductive core identical radii.

III. Experimental Apparatus

This conductivity measurement technique does not involve the use of elaborate apparatus. The only strict requirements imposed on the apparatus' design are that the axial magnetic field be uniform in the neighborhood of the sensor coil, that the magnetic field itself be relatively weak, so as not to appreciably disturb the state of the plasma by the induction of strong eddy currents, and that the magnetic field be constant throughout the measurement.

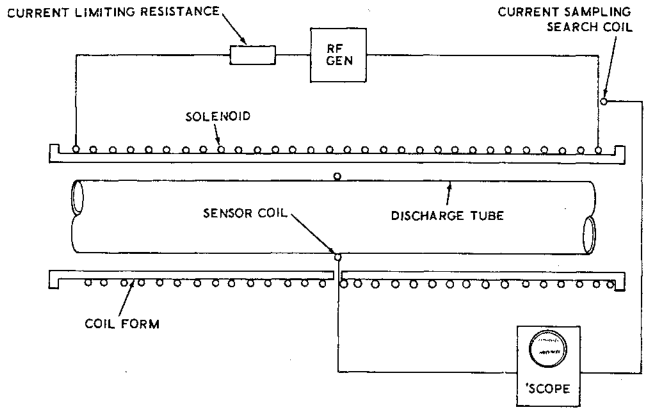

Fig. 1 Apparatus used in the measurement of the electrical conductivity of a plasma.

Any field-producing device that meets the preceding requirements will suffice for the conductivity measurement provided that the appropriate substitution for B_0 be made in Eq. (6). These conditions are fulfilled with apparatus as shown in Fig. 1. The solenoid, wound on a suitable form such that its diameter is much less than its length, is isolated from the rf generator, supplying its current by means of a large resistance, or some equivalent device. The solenoid is mounted coaxially with the plasma discharge tube. The sensor coil is mounted on the discharge tube in the central plane of the solenoid.

The apparatus is calibrated by observing the magnetic flux through the sensor coil when it contains a known conductor. A variable frequency oscillator drives the solenoid, which produces the alternating magnetic field. The flux is obtained as a function of $2^{1/2}R/\delta$ by varying both the conductivity of the calibration sample and the frequency of the applied field. This approach is necessary because of the difficulty with which conductivity standards in the range of interest (i.e., 10–1000 mho/m) are obtained. Absolute flux measurements as a function of $2^{1/2}R/\delta$ should be predicted by Eq. (5), whereas relative flux measurements should be predicted by Eqs. (7) or (8).

The results of a calibration that has been described elsewhere[15] are presented in Fig. 2. In this case, the normalizing flux Φ_0 was observed at the various frequencies with a copper cylinder positioned in the sensor coil.

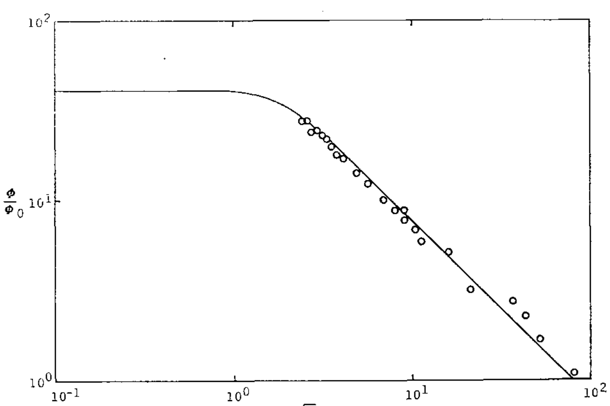

Fig. 2 Theoretical prediction and experimental observation of normalized magnetic flux as a function of $2^{1/2}R/\delta$. The normalizing conductor in this case is copper.

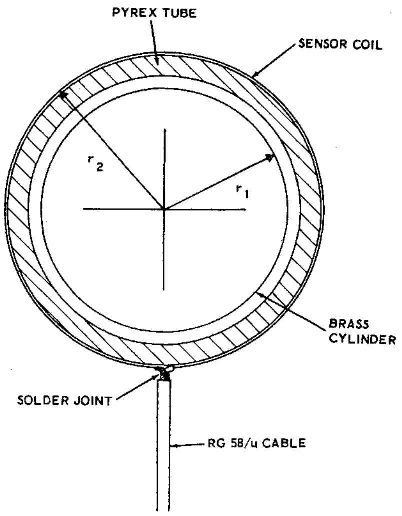

Fig. 3 Cross section of discharge tube showing sensor coil and brass calibration cylinder. The area A_1 of the brass cylinder normal to the external flux is given by πr_1^2; the corresponding area A_2 of the sensor coil is πr_2^2.

The conductivity measurement technique is most sensitive [Ref. 14, p. 165, Fig. 32; this same result can be obtained by differentiation of Eq. (5)] when $2^{1/2}R/\delta \approx 2.5$. In most applications of laboratory proportions, therefore, the operating frequency for maximum sensitivity is in the low megacycle range. This presents a simple method for testing the proper operation of the measurement apparatus prior to its use.

The signal induced on the sensor coil by the time varying axial magnetic field of the solenoid is recorded. A brass or copper cylinder with a diameter nearly equal to that of the discharge tube is then placed in the discharge tube, and the resulting attenuated flux signal is recorded. Since brass or copper acts like an infinite conductor to the megacycle field, a nearly complete exclusion of the magnetic flux from the conductor is predicted from Eq. (7) or (8).

The ratio of the attenuated flux and the unattenuated flux is then formed and the areal correction applied. This ratio is then compared with the right-hand side of Eq. (8). Agreement to about 5% is typical in the author's experience; a greater deviation from theory is probably indicative of stray rf pickup, poor isolation of the rf generator, or an incorrect value for the reference material's conductivity.

The determination of the conductivity of a plasma follows the preceding procedure. The flux ratio is formed, and the quantity $2^{1/2}R/\delta$ is determined from Eqs. (7) or (8) by successive approximation. The conductivity is then obtained from the skin-depth relation.

Appendix: A Sample Measurement

As an example of the use of this technique, a determination of the electrical conductivity of the exhaust gas of a sample of burning solid rocket fuel will be described. The fuel was positioned in a small metal chamber attached to the end of a pyrex vacuum line. The hot exhaust gas expanded through the vacuum line into a large metal dump tank. The solenoid providing the magnetic field was coaxial with the pyrex vacuum line.

The solenoid was wound on a Micarta form, which was approximately 24 in. in length by 3.5 in. in diameter. It had thirty turns of # 20 wire. The constant current through the solenoid was supplied by a crystal controlled 14.3 Mc transmitter operated at a low power level and limited by a large resistance inserted in the transmission line coupling the transmitter to the solenoid. The current was sampled by displaying the signal from a loop of wire loosely coupled to the solenoid on one channel of a Tektronix 555 oscilloscope.

The sensor coil was formed from one turn of the center conductor of a length of RG 58/U cable tightly wound about the pyrex vacuum line and soldered to its shield as in Fig. 3. It was located at the center of the solenoid.

The proper operation of the apparatus was checked in a simple exercise. A brass cylinder was placed in the pyrex tube, and the sensor coil signal before and after insertion was photographed. Since at 14.3 Mc brass acts like a nearly perfect conductor (i.e., $\delta = 3.5 \times 10^{-5}$ m), a complete exclusion of flux from the brass' volume was anticipated. The signals after and before insertion should be given by

$$\frac{\Phi}{\Phi_0} \approx \frac{B_0(A_2 - A_1)}{B_0 A_2} = 1 - \frac{r_1^2}{r_2^2} \tag{A1}$$

The radius of the brass cylinder r_1 equaled 2.25 cm, and the radius of the sensor coil r_2 was 2.99 cm. The anticipated magnitude of the flux reduction was calculated as 43.3% (i.e., the magnitude of the attenuated signal is 43.3% as great as the magnitude of the unattenuated signal).

Then

$$\frac{\Phi_{\text{brass}}}{\Phi_{\text{air}}} = \frac{\text{unattenuated signal} - \text{attenuated signal}}{\text{unattenuated signal}}$$

$$= \frac{102 - 56}{102} = 0.451 \text{ (arbitrary units)}$$

The deviation from theory

$$\frac{45.1 - 43.3}{43.3} \times 100 = 4.15\%$$

is well within the limits imposed by the apparatus.

The measurement of the exhaust conductivity was made in a fashion analogous to the exercise just described. The sensor coil was positioned far enough downstream from the nozzle so that it was safe to assume that the radius of the exhaust was equal to the inside radius of the vacuum line ($r = 2.54$ cm), and that the plasma was homogeneous at the measuring station. The percentage of area increase over the brass calibration is then

$$[(2.54)^2 - (2.25)^2]/(2.25)^2 \times 100 = 27.5\%$$

In a case where the plasma conductivity is sufficiently high to completely exclude the flux from its volume, a decrease in signal of (43.3% + 27.5%) = 70.8% is expected. Thus 70.8% of the unattenuated signal is due to the (plasma) inner area; 29.2% of the unattenuated signal is due to the outer area.

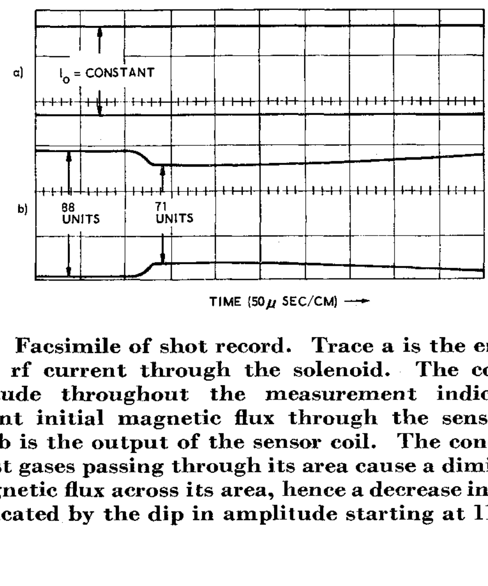

Fig. 4 Facsimile of shot record. Trace a is the envelope of the rf current through the solenoid. The constant amplitude throughout the measurement indicates a constant initial magnetic flux through the sensor coil. Trace b is the output of the sensor coil. The conductive exhaust gases passing through its area cause a diminution of magnetic flux across its area, hence a decrease in signal, as indicated by the dip in amplitude starting at 110 μsec.

A facsimile of a typical shot record is presented in Fig. 4, where the envelopes of the current and flux signals are shown. The oscilloscope was triggered by the firing signal. The delay between the triggering of the sweep and the arrival of the conductivity signal represents the time required for the fuel to ignite and for the exhaust to propagate down the vacuum line to the position of the sensor coil. There is some question as to the resolution in time of this signal; hence, no velocity measurement is reported.

The unattenuated shot signal has an arbitrary amplitude of 88 units (measured on the photograph with dividers and a machine divided scale). The part of this signal due to the inner (plasma) area is 70.8% of this, or 62.2 units. At maximum diminution (i.e., maximum conductivity) the shot signal has an amplitude of 71 units, a reduction of 17 units. Since the entire diminution of flux takes place within the inner area, the flux ratio is given by

$$\frac{\Phi_{\text{shot}}}{\Phi_{\text{vacuum}}} = \frac{62.2 - 17}{62.2} = \frac{45.2}{62.2} = 0.726$$

From the theory

$$\frac{\Phi}{\Phi_0} = 0.726 = 2\left(\frac{\delta}{2^{1/2}R}\right)\frac{M_1(2^{1/2}R/\delta)}{M_0(2^{1/2}R/\delta)} \tag{A2}$$

which is solved by successive approximation as follows: let

$$x = 2^{1/2}R/\delta$$

then

$$0.363\ x = M_1(x)/M_0(x)$$

To a value as close as the tables (Ref. 14, pp. 227–228) of M go, without interpolation, $x = 2^{1/2}R/\delta = R(\sigma\mu_0\omega)^{1/2} \approx 2.5$. Therefore,

$$\sigma = (2.5)^2/R^2\mu_0\omega \tag{A3}$$

where

$$R = 2.54 \times 10^{-2}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/m}$$

$$\omega = 2\pi f = 9 \times 10^7 \text{ rad/sec}$$

Substitution of these values in Eq. (A3) yields a value for the conductivity at its maximum: $\sigma \approx 86$ mho/m.

In this particular example, the conductivity of the exhaust gas may be somewhat higher than usual because of the seeding effect of the melting of the metal orifice through which the hot gas has passed. This measurement therefore may not be typical of the conductivities to be expected in the deflagration of all solid rocket fuel.

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